Surjectivity in Fréchet Spaces

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Theorem of Nash and Moser is an useful tool for studying solvability of certain nonlinear problems with infinitely smooth data. What is important for applications is surjectivity. The work [1], which was one of our inspirations, shows that certain surjectivity can be proved for functions which are only Gâteaux differentiable. However, the estimate obtained there is for one chosen norm of the Fréchet space (more precisely, for linear combination of the norms, but with non-canonic coefficients), while in the original Nash-Moser Theorem all seminorms are estimated.

In [2] we further the above development by proving surjectivity result for multi-valued maps with estimates for *all* semi-norms. It readily renders to the case of Gâteaux differentiable function:

Theorem 1. *Let $X = \cap_0^\infty (X_n, \|\cdot\|_n)$ and $Y = \cap_0^\infty (Y_n, |\cdot|_n)$ be CI spaces. Let $f : X \to Y$ be continuous, Gâteaux differentiable and such that $f(0) = 0$.*

Assume that there are $c > 0$ and $d \in \{0\} \cup \mathbb{N}$, such that for each $x \in X$ and $v \in Y$

$$\exists u \in X : \; f'(x)u = v \text{ and } \|u\|_n \le c|v|_{n+d}, \quad \forall n \ge 0.$$

Then for each $y \in Y$ there is $x \in X$ such that

$$f(x) = y \text{ and } \|x\|_n \le c|y|_{n+d}, \quad \forall n \ge 0.$$

CI space (from C-Infinity) is called any $X = \cap_0^\infty X_n$, where $(X_n, \|\cdot\|_n)$'s are nested Banach spaces: $X_{n+1} \subset X_n$, such that the identity operator from X_{n+1} into X_n is compact. $C^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is compact domain, is CI space.

Comparing this statement to [1, Theorem 1], we note that our assumptions on X are much more restrictive but the most important cases of infinitely smooth functions on compacts are covered. On the other hand, we do not bound the norms of the derivatives and do not require the existence of left inverse of these derivatives. The most significant is that we have estimates for *all* norms simultaneously.

References

[1] I. Ekeland, An inverse function theorem in Fréchet spaces, Ann. Inst. H. Poincaré, C (2011), volume 28, issue 1, pp. 91–105.

[2] M. Ivanov and N. Zlateva, Surjectivity in Fréchet spaces, arXiv:1805.07055 [math.FA], 2018 (https://arxiv.org/pdf/1805.07055.pdf).